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Non-local boundary value problems for impulsive fractional integro-differential equations in Banach spaces

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Abstract

In this study, we establish some conditions for existence and uniqueness of the solutions to semilinear fractional impulsive integro-differential evolution equations with non-local conditions by using Schauder's fixed point theorem and the contraction mapping principle.

MSC: 26A33; 34A37

Keywords: boundary value problem; Caputo type fractional derivative; existence and uniqueness; fixed point theorem; impulsive integro-differential equation; nonlocal condition

1 Introduction

The topic of fractional differential equations has received a great deal of attention from many scientists and researchers during the past decades; see, for instance, [1–7]. This is mostly due to the fact that fractional calculus provides an efficient and excellent instrument to describe many practical dynamical phenomena which arise in engineering and science such as physics, chemistry, biology, economy, viscoelasticity, electrochemistry, electromagnetic, control, porous media; see [8–13]. Moreover, many researchers study the existence of solutions for fractional differential equations; see [14–16] and the references therein.

In particular, several authors have considered a nonlocal Cauchy problem for abstract evolution differential equations having fractional order. Indeed, the nonlocal Cauchy problem for abstract evolution differential equations was studied by Byszewski [17, 18] initially. Afterwards, many authors [19–21] discussed the problem for different kinds of nonlinear differential equations and integrodifferential equations including functional differential equations in Banach spaces. Balachandran *et al.* [22, 23] established the existence of solutions of quasilinear integrodifferential equations with nonlocal conditions. N'Guérékata [24] and Balachandran and Park [25] researched the existence of solutions of fractional abstract differential equations with a nonlocal initial condition. Ahmad [26] obtained some existence results for boundary value problems of fractional semilinear evolution equations. Recently, Balachandran and Trujillo [27] have investigated the nonlocal Cauchy problem for nonlinear fractional integrodifferential equations in Banach spaces.

On the other hand, the theory of impulsive differential equations for integer order has emerged in mathematical modeling of phenomena and practical situations in both physical and social sciences in recent years. One can see a significant development in impulsive theory. We refer the readers to [28–31] for the general theory and applications of impulsive differential equations. Besides, some researchers (see [32–35] and the references therein) have addressed the theory of boundary value problems for impulsive fractional differential equations.

However, only a few studies were concerned with the Cauchy problem for impulsive evolution differential equations of fractional order; see [36–38]. Further, in [38], Balachandran *et al.* studied the existence of solutions for fractional impulsive integrodifferential equations of the following type:

$$\begin{aligned}
&{}^{C}D^{q}u(t) = A(t,u)u(t) + f\left(t, u(t), \int_0^t h\bigl(t,s,u(s)\bigr)\,ds\right),\\
&\Delta u(t_k) = I_k\bigl(u\bigl(t_k^-\bigr)\bigr),\\
&u(0) + g(u) = u_0,
\end{aligned}$$

where $0 \le t \le T$ and $0 < q < 1$, by using the contraction mapping principle.

Motivated by the aforementioned works, in this paper, we deal with the existence and uniqueness of solutions for a boundary value problem (BVP), for the following impulsive fractional semilinear integro-differential equation with nonlocal conditions:

$$\begin{cases}
{}^{C}D^{q}u(t) = A(t)u(t) + f(t,u(t),\int_0^t k(t,s,u(s))\,ds), \quad t \in J := [0,1], t \neq t_k,\\
\Delta u(t_k) = I_k(u(t_k^-)), \qquad \Delta u'(t_k) = I_k^*(u(t_k^-)), \quad k = 1,2,\dots,p,\\
\alpha u(0) + \beta u'(0) = g_1(u), \qquad \alpha u(1) + \beta u'(1) = g_2(u),
\end{cases} \tag{1.1}$$

where $1 < q < 2$, ${}^{C}D^{\alpha}$ is the Caputo fractional derivative, $A(t)$ is a bounded linear operator on a Banach space X, $f \in C(J \times X \times X, X)$, $k \in C(\mathbb{k} \times X, X)$, $I_k, I_k^* \in C(X,X)$, $g_1, g_2 : PC(J,X) \to X$ ($PC(J,X)$ will be defined in the next section),

$$\Delta u(t_k) = u\bigl(t_k^+\bigr) - u\bigl(t_k^-\bigr)$$

with

$$u\bigl(t_k^+\bigr) = \lim_{h\to 0^+} u(t_k + h), \qquad u\bigl(t_k^-\bigr) = \lim_{h\to 0^-} u(t_k + h)$$

and $\Delta u'(t_k)$ has a similar meaning for $u'(t)$, $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = 1$, and $\alpha, \beta \ge 0$. Here $\mathbb{k} = \{(t,s) : 0 \le s \le t \le 1\}$. For brevity, let us take $Ku(t) = \int_0^t k(t,s,u(s))\,ds$.

Meanwhile, nonlinear functions f of this type with the integral term k occur in mathematical problems that are concerned with the heat flow in materials having memory and viscoelastic problems; see [39]. Also, as indicated in [40, 41], nonlocal conditions can be more useful than standard conditions to describe physical phenomena. For example, in [41], the author described the diffusion phenomenon of a small amount of gas in a transparent tube by using the formula

$$g(u) = \sum_{i=1}^{m} \eta_i u(\xi_i),$$

where η_i, $i = 1,2,\dots,m$ are given constants and $0 < \xi_1 < \xi_2 < \cdots < \xi_m < T$.

Note that in this work, to the best of our knowledge, it is the first time that a general boundary value problem for impulsive semilinear evolution integrodifferential equations of fractional order $1 < q < 2$ with nonlocal conditions has been considered.

The rest of this paper is organized as follows. In Section 2, we present some notations and preliminary results about fractional calculus and differential equations to be used in the following sections. In Section 3, we discuss some existence and uniqueness results for solutions of BVP (1.1). Namely, the first result is based on Schauder's fixed point theorem and the second one is based on Banach's fixed point theorem. Finally, we shall give an illustrative example for our results.

2 Preliminaries

In order to model the real world application, the fractional differential equations are considered by using the fractional derivatives. There are many different starting points for the discussion of classical fractional calculus; see, for example, [42]. One can begin with a generalization of repeated integration. If $f(t)$ is absolutely integrable on $[0, b)$, it can be found [42, 43]

$$\begin{aligned}\int_0^t dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_3} dt_2 \int_0^{t2} f(t_1)\,dt_1 &= \frac{1}{(n+1)!}\int_0^t (t-t_1)^{n-1} f(t_1)\,dt_1 \\ &= \frac{1}{(n+1)!} t^{n-1} * f(t),\end{aligned}$$

where $n = 1, 2, \dots$ and $0 \leq t \leq b$. On writing $\Gamma(n) = (n-1)!$, an immediate generalization in the form of the operation I^α defined for $\alpha > 0$ is

$$\begin{aligned}\bigl(I^\alpha f\bigr)(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-t_1)^{\alpha-1} f(t_1)\,dt_1 \\ &= \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t), \quad 0 \leq t < b, \end{aligned} \tag{2.1}$$

where $\Gamma(\alpha)$ is the gamma function and $t^{\alpha-1} * f(t) = \int_0^t f(t-t_1)^{\alpha-1}(t_1)\,dt_1$ is called the convolution product of $t^{\alpha-1}$ and $f(t)$. Now Eq. (2.1) is known as a fractional integral of order α for the function $f(t)$.

Next, we give some basic definitions and properties of fractional calculus theory used in this paper; see [1, 4, 28, 31, 32].

Let $J_0 = [0, t_1]$, $J_1 = (t_1, t_2], \dots, J_{k-1} = (t_{k-1}, t_k]$, $J_k = (t_k, t_{k+1}]$, and $J' := [0, T]\backslash\{t_1, t_2, \dots, t_p\}$, then we define the set of functions as follows:

$PC(J, X) = \{u : J \to X : u \in C((t_k, t_{k+1}], X), k = 0, 1, 2, \dots, p$ and there exist $u(t_k^+)$ and $u(t_k^-), k = 1, 2, \dots, p$ with $u(t_k^-) = u(t_k)\}$ and

$PC^1(J, X) = \{u \in PC(J, X), u' \in C((t_k, t_{k+1}], X), k = 0, 1, 2, \dots, p$ and there exist $u'(t_k^+)$ and $u'(t_k^-), k = 1, 2, \dots, p$ with $u'(t_k^-) = u'(t_k)\}$ which is a Banach space with the norm

$$\|u\| = \sup_{t\in J}\bigl\{\|u\|_{PC}, \|u'\|_{PC}\bigr\} \quad \text{where } \|u\|_{PC} := \sup\bigl\{|u(t)| : t \in J\bigr\}.$$

Now, $B(X)$ denotes the Banach space of bounded linear operators from X into X with the norm $\|A\|_{B(X)} = \sup\{\|A(u)\| : \|u\| = 1\}$.

Definition 1 [1, 4] The fractional (arbitrary) order integral of the function $h \in L^1(J, R_+)$ of order $q \in R_+$ is defined by

$$I^q_{0^+} h(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)\, ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2 [1, 4] For a function h given on the interval J, the Caputo-type fractional derivative of order $q > 0$ is defined by

$${}^C D^q_{0^+} h(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} h^{(n)}(s)\, ds, \quad n = [q] + 1,$$

where the function $h(t)$ has absolutely continuous derivatives up to order $(n-1)$.

Lemma 1 [1] *Let $q > 0$, then the differential equation*

$${}^C D^q h(t) = 0$$

has the following solution:

$$h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \quad c_i \in R, i = 0, 1, 2, \ldots, n-1, n = [q] + 1.$$

Lemma 2 [14] *Let $q > 0$, then*

$$I^{q\,C} D^q h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$$

for some $c_i \in R$, $i = 0, 1, 2, \ldots, n-1$, $n = [q] + 1$.

Now, by using the Kronecker convolution product, see [7], the fractional integral becomes

$$\big(I^\alpha f\big)(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} * f(x) \simeq \xi^T \frac{1}{\Gamma(\alpha)} \big\{x^{\alpha-1} * \phi_m(x)\big\}. \tag{2.2}$$

Thus, if $x^{\alpha-1} * \phi_m(x)$ can be integrated, then expanded in block pulse functions, the fractional integral is solved via the block pulse functions operational matrix as follows:

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-t_1)^{\alpha-1} \phi_m(t_1)\, dt_1 \simeq F_\alpha \phi_m(t),$$

where

$$\psi_m(t) = \begin{cases} 1 & (\frac{m-1}{i})b \le t < (\frac{m}{i})b, \\ 0 & \text{elsewhere} \end{cases}$$

for $m = 1, 2, \ldots, i$ and

$$F_\alpha = \left(\frac{b}{m}\right)^\alpha \frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \xi_2 & \xi_3 & \cdots & \xi_m \\ 0 & 1 & \xi_2 & \cdots & \xi_{m-1} \\ 0 & 0 & 1 & \cdots & \xi_{m-2} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix};$$

see [7].

Now, we need the following lemma for our study.

Lemma 3 *Let $1 < q < 2$ and $h : J \to X$ be continuous. A function $u(t)$ is a solution of the fractional integral equation*

$$u(t) = \begin{cases} \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} h(s)\,ds + (\frac{\beta}{\alpha} - t)[\int_{t_k}^1 \frac{(1-s)^{q-1}}{\Gamma(q)} h(s)\,ds + \frac{\beta}{\alpha}\int_{t_k}^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} h(s)\,ds \\ \quad + \sum_{i=1}^k (\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)} h(s)\,ds + I_i(u(t_i^-))) \\ \quad + \sum_{i=1}^k (\frac{\beta}{\alpha} + 1 - t_k)(\int_{t_{i-1}}^{t_i} \frac{(t_1-s)^{q-2}}{\Gamma(q-1)} h(s)\,ds + I_i^*(u(t_i^-))) \\ \quad + \frac{1}{\alpha}(g_1(u) - g_2(u))] + \frac{g_1(u)}{\alpha}, & t \in J_0, \\ \int_{t_k}^t \frac{(t-s)^{q-1}}{\Gamma(q)} h(s)\,ds + \sum_{i=1}^k (\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)} h(s)\,ds + I_i(u(t_i^-))) \\ \quad + \sum_{i=1}^k (t - t_i)\int_{t_{i-1}}^{t_i} (\frac{(t_i-s)^{q-2}}{\Gamma(q-1)} h(s)\,ds + I_i^*(u(t_i^-))) \\ \quad + (\frac{\beta}{\alpha} - t)[\int_{t_k}^1 \frac{(1-s)^{q-1}}{\Gamma(q)} h(s)\,ds + \frac{\beta}{\alpha}\int_{t_k}^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} h(s)\,ds \\ \quad + \sum_{i=1}^k (\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)} h(s)\,ds + I_i(u(t_i^-))) \\ \quad + \sum_{i=1}^k (\frac{\beta}{\alpha} + 1 - t_k)(\int_{t_{i-1}}^{t_i} \frac{(t_1-s)^{q-2}}{\Gamma(q-1)} h(s)\,ds + I_i^*(u(t_i^-))) \\ \quad + \frac{1}{\alpha}(g_1(u) - g_2(u))] + \frac{g_1(u)}{\alpha}, & t \in J_k \end{cases} \tag{2.3}$$

if and only if $u(t)$ is a solution of the fractional BVP

$$\begin{aligned} &{}^C D^q u(t) = h(t), \quad t \in J' \\ &\Delta u(t_k) = I_k\big(u(t_k^-)\big), \qquad \Delta u'(t_k) = I_k^*\big(u(t_k^-)\big), \\ &\alpha u(0) + \beta u'(0) = g_1(u), \qquad \alpha u(1) + \beta u'(1) = g_2(u), \end{aligned} \tag{2.4}$$

where $k = 1, 2, \ldots, p$.

Proof Let u be the solution of (2.4). If $t \in J_0$, then Lemma 2 implies that

$$\begin{aligned} u(t) &= I^q h(t) - c_0 - c_1 t = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} h(s)\,ds - c_0 - c_1 t, \\ u'(t) &= \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)} h(s)\,ds - c_1 \end{aligned}$$

for some $c_0, c_1 \in R$.

Applying the boundary condition $\alpha u(0) + \beta u'(0) = g_1(u)$ for $t \in J_0$, we find that

$$u(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} h(s)\,ds + c_1\left(\frac{\beta}{\alpha} - t\right) + \frac{g_1(u)}{\alpha}. \tag{2.5}$$

If $t \in J_1$, then Lemma 2 implies that

$$\begin{aligned} u(t) &= \int_{t_1}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} h(s)\,ds - d_0 - d_1(t-t_1), \\ u'(t) &= \int_{t_1}^{t} \frac{(t-s)^{q-2}}{\Gamma(q-1)} h(s)\,ds - d_1 \end{aligned}$$

for some $d_0, d_1 \in R$. Thus, we have

$$\begin{aligned} u(t_1^-) &= \int_0^{t_1} \frac{(t_1-s)^{q-1}}{\Gamma(q)} h(s)\,ds + c_1\left(\frac{\beta}{\alpha} - t_1\right) + \frac{g_1(u)}{\alpha}, \qquad u(t_1^+) = -d_0, \\ u'(t_1^-) &= \int_0^{t_1} \frac{(t_1-s)^{q-2}}{\Gamma(q-1)} h(s)\,ds - c_1, \qquad u'(t_1^+) = -d_1. \end{aligned}$$

In the view of

$$\Delta u(t_1) = u(t_1^+) - u(t_1^-) = I_1(u(t_1^-)) \quad \text{and} \quad \Delta u'(t_1) = u'(t_1^+) - u'(t_1^-) = I_1^*(u(t_1^-)),$$

we have

$$\begin{aligned} -d_0 &= \int_0^{t_1} \frac{(t_1-s)^{q-1}}{\Gamma(q)} h(s)\,ds + c_1\left(\frac{\beta}{\alpha} - t_1\right) + \frac{g_1(u)}{\alpha} + I_1(u(t_1^-)), \\ -d_1 &= \int_0^{t_1} \frac{(t_1-s)^{q-2}}{\Gamma(q-1)} h(s)\,ds - c_1 + I_1^*(u(t_1^-)). \end{aligned}$$

Hence,

$$\begin{aligned} u(t) &= \int_{t_1}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} h(s)\,ds + \int_0^{t_1} \frac{(t_1-s)^{q-1}}{\Gamma(q)} h(s)\,ds \\ &\quad + (t-t_1)\left[\int_0^{t_1} \frac{(t_1-s)^{q-2}}{\Gamma(q-1)} h(s)\,ds + I_1^*(u(t_1^-))\right] \\ &\quad + I_1(u(t_1^-)) + c_1\left(\frac{\beta}{\alpha} - t\right) + \frac{g_1(u)}{\alpha}, \quad \text{for } t \in J_1. \end{aligned}$$

By repeating the process, for $t \in J_k$, we have

$$\begin{aligned} u(t) &= \int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} h(s)\,ds + \sum_{i=1}^{k}\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)} h(s)\,ds + I_i(u(t_i^-))\right] \\ &\quad + \sum_{i=1}^{k}(t-t_i)\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)} h(s)\,ds + I_i^*(u(t_i^-))\right] \\ &\quad + c_1\left(\frac{\beta}{\alpha} - t\right) + \frac{g_1(u)}{\alpha}. \end{aligned} \tag{2.6}$$

Now, applying the boundary condition

$$\alpha u(1) + \beta u'(1) = g_2(u),$$

we find that

$$\begin{aligned}
c_1 &= \int_{t_k}^{1} \frac{(1-s)^{q-1}}{\Gamma(q)} h(s)\,ds + \frac{\beta}{\alpha} \int_{t_k}^{1} \frac{(1-s)^{q-2}}{\Gamma(q-1)} h(s)\,ds \\
&\quad + \sum_{i=1}^{k} \biggl[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)} h(s)\,ds + I_i\bigl(u\bigl(t_i^-\bigr)\bigr) \biggr] \\
&\quad + \sum_{i=1}^{k} \biggl(\frac{\beta}{\alpha} + 1 - t_k \biggr) \biggl[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)} h(s)\,ds + I_i^*\bigl(u\bigl(t_i^-\bigr)\bigr) \biggr] \\
&\quad + \frac{1}{\alpha} \bigl[g_1(u) - g_2(u) \bigr].
\end{aligned}$$

Substituting the value of c_1 in (2.5) and (2.6), we obtain Eq. 2.3.

Conversely, if we assume that u satisfies the impulsive fractional integral equation (2.3), then by direct computation, we can easily see that the solution given by (2.3) satisfies (2.4). Thus, the proof of Lemma 3 is complete. □

3 Main results

Definition 3 A function $u \in PC^1(J, X)$ with its q-derivative existing on J' is said to be a solution of (1.1) if u satisfies the equation

$$^{C}D^{q} u(t) = A(t)u(t) + f\bigl(t, u(t), Ku(t)\bigr)$$

on J' and satisfies the conditions

$$\begin{aligned}
&\Delta u(t_k) = I_k\bigl(u\bigl(t_k^-\bigr)\bigr), \qquad \Delta u'(t_k) = I_k^*\bigl(u\bigl(t_k^-\bigr)\bigr), \\
&\alpha u(0) + \beta u'(0) = g_1(u), \qquad \alpha u(1) + \beta u'(1) = g_2(u).
\end{aligned}$$

Now, we define the operator $T : PC^1(J, X) \to PC^1(J, X)$ by

$$Tu(t) = \begin{cases}
\int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} (A(s)u(s) + f(s, u(s), Ku(s)))\,ds \\
\quad + \sum_{i=1}^{k} (\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)} (A(s)u(s) + f(s, u(s), Ku(s)))\,ds + I_i(u(t_i^-))) \\
\quad + \sum_{i=1}^{k} (t - t_i) \\
\quad \times (\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)} (A(s)u(s) + f(s, u(s), Ku(s)))\,ds + I_i^*(u(t_i^-))) \\
\quad + (\frac{\beta}{\alpha} - t)[\int_{t_k}^{1} \frac{(1-s)^{q-1}}{\Gamma(q)} (A(s)u(s) + f(s, u(s), Ku(s)))\,ds \\
\quad + \frac{\beta}{\alpha} \int_{t_k}^{1} \frac{(1-s)^{q-2}}{\Gamma(q-1)} (A(s)u(s) + f(s, u(s), Ku(s)))\,ds \\
\quad + \sum_{i=1}^{k} (\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)} (A(s)u(s) + f(s, u(s), Ku(s)))\,ds + I_i(u(t_i^-))) \\
\quad + \sum_{i=1}^{k} (\frac{\beta}{\alpha} + 1 - t_k) \\
\quad \times (\int_{t_{i-1}}^{t_i} \frac{(t_1-s)^{q-2}}{\Gamma(q-1)} (A(s)u(s) + f(s, u(s), Ku(s)))\,ds + I_i^*(u(t_i^-))) \\
\quad + \frac{1}{\alpha}(g_1(u) - g_2(u))] + \frac{g_1(u)}{\alpha}, \quad t \in J_k.
\end{cases} \tag{3.1}$$

Clearly, the fixed points of the operator T are the solutions of problem (1.1). To begin with, we need the following assumptions to prove the existence and uniqueness of a solution of the integral equation (2.3) which satisfies BVP (1.1):

(A1) $A : X \to X$ is a continuous bounded linear operator and there exists a constant $A_1 > 0$ such that $\|A(u)\|_{B(X)} \le A_1$ for all $u \in X$;
(A2) The function $f : J \times X \times X \to X$ is continuous and there exists a constant $M_1 > 0$ such that $M_1 = \max_{s\in J}\{f(s,u(s),Ku(s)), u \in X\}$;
(A3) $I_k, I_k^* : X \to X$ are continuous and there exist constants $M_2 > 0$ and $M_3 > 0$ such that $\|I_k(u)\| \le M_2$, $\|I_k^*(u)\| \le M_3$ for each $u \in X$ and $k = 1,2,\dots,p$;
(A4) There exist constants$G_i > 0$ and $g_i : PC^1(J,X) \to X$ are continuous functions such that $\|g_i(u)\| \le G_i$, $i = 1,2$;
(A5) There exists a constant $L_1 > 0$ such that

$$\|f(t,u_1,v_1) - f(t,u_2,v_2)\| \le L_1\bigl(\|u_1 - u_2\| + \|v_1 - v_2\|\bigr),$$

$\forall t \in J$, and $u_1,u_2,v_1,v_2 \in X$;
(A6) $k : \mathbb{k} \times X \to X$ is continuous and there exists a constant $L_2 > 0$ such that

$$\|k(t,s,u) - k(t,s,v)\| \le L_2\|u - v\|$$

for all $u,v \in X$;
(A7) There exist constants $L_3 > 0$, $L_4 > 0$ such that $\|I_k(u) - I_k(v)\| \le L_3\|u - v\|$, $\|I_k^*(u) - I_k^*(v)\| \le L_4\|u - v\|$ for each $u,v \in X$ and $k = 1,2,\dots,p$;
(A8) There exist constants $b_i > 0$ such that $\|g_i(u) - g_i(v)\| \le b_i\|u - v\|$, $i = 1,2$.

The following are the main results of this paper. Our first result relies on Schauder's fixed point theorem which gives an existence result for solutions of BVP (1.1).

Theorem 1 *Assume that the assumptions* (A1)-(A4) *hold. Then BVP* (1.1) *has at least one solution on* J.

Proof In order to show the existence of a solution of BVP (1.1), we need to transform BVP (1.1) to a fixed point problem by using the operator T in (3.1). Now, we shall use Schauder's fixed point theorem to prove T has a fixed point which is then a solution of BVP (1.1). First, let us define $B_r = \{u \in PC^1(J) : \|u\| \le r\}$ for any $r > 0$. Then it is clear that the set B_r is a closed, bounded and convex. The proof will be given in several steps.

Step 1: T is continuous.

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $PC(J)$. Then

$$\begin{aligned}
&\bigl|(Tu_n)(t) - (Tu)(t)\bigr| \\
&\quad\le \int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)}\bigl(|A(s)||u_n(s) - u(s)| \\
&\qquad + \bigl|f\bigl(s,u_n(s),Ku_n(s)\bigr) - f\bigl(s,u(s),Ku(s)\bigr)\bigr|\bigr)\,ds \\
&\qquad + \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)}\bigl(|A(s)||u_n(s) - u(s)| \\
&\qquad + \bigl|f\bigl(s,u_n(s),Ku_n(s)\bigr) - f\bigl(s,u(s),Ku(s)\bigr)\bigr| + \bigl|I_i\bigl(u_n(t_i^-)\bigr) - I_i\bigl(u(t_i^-)\bigr)\bigr|\bigr)\,ds \\
&\qquad \times \sum_{i=1}^{k}(t - t_i)\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)}\bigl(|A(s)||u_n(s) - u(s)|
\end{aligned}$$

$$
\begin{aligned}
&+ \left|f\left(s,u_n(s),Ku_n(s)\right) - f\left(s,u(s),Ku(s)\right)\right| \\
&+ \left|I_i^*\left(u_n\left(t_i^-\right)\right) - I_i^*\left(u\left(t_i^-\right)\right)\right|\Big)\, ds \\
&+ \left|\frac{\beta}{\alpha} - t\right| \Bigg[\int_{t_k}^{1} \frac{(1-s)^{q-1}}{\Gamma(q)} \Big(\left|A(s)\right|\left|u_n(s) - u(s)\right| \\
&+ \left|f\left(s,u_n(s),Ku_n(s)\right) - f\left(s,u(s),Ku(s)\right)\right|\Big)\, ds \\
&+ \frac{\beta}{\alpha} \int_{t_k}^{1} \frac{(1-s)^{q-2}}{\Gamma(q-1)} \Big(\left|A(s)\right|\left|u_n(s) - u(s)\right| \\
&+ \left|f\left(s,u_n(s),Ku_n(s)\right) - f\left(s,u(s),Ku(s)\right)\right|\Big)\, ds \\
&+ \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)} \Big(\left|A(s)\right|\left|u_n(s) - u(s)\right| \\
&+ \left|f\left(s,u_n(s),Ku_n(s)\right) - f\left(s,u(s),Ku(s)\right)\right| + \left|I_i\left(u_n\left(t_i^-\right)\right) - I_i\left(u\left(t_i^-\right)\right)\right|\Big)\, ds \\
&+ \sum_{i=1}^{k} \left(\frac{\beta}{\alpha} + 1 - t_k\right) \int_{t_{i-1}}^{t_i} \frac{(t_1-s)^{q-2}}{\Gamma(q-1)} \Big(\left|A(s)\right|\left|u_n(s) - u(s)\right| \\
&+ \left|f\left(s,u_n(s),Ku_n(s)\right) - f\left(s,u(s),Ku(s)\right)\right| + \left|I_i^*\left(u_n\left(t_i^-\right)\right) - I_i^*\left(u\left(t_i^-\right)\right)\right|\Big)\, ds \\
&+ \frac{1}{\alpha}\Big(\left|g_1(u_n) - g_1(u)\right| + \left|g_2(u_n) - g_2(u)\right|\Big)\Bigg] + \frac{1}{\alpha}\left|g_1(u_n) - g_1(u)\right|.
\end{aligned}
$$

Since A is a continuous operator and f, g, I, I^* are continuous functions, we have $\|Tu_n - Tu\| \to 0$ as $n \to \infty$.

Step 2: T maps bounded sets into bounded sets.

Now, it is enough to show that there exists a positive constant l such that $\|Tu\| \le l$ for each $u \in B_r$. Then we have, for each $t \in J$,

$$
\begin{aligned}
\left|(Tu)(t)\right| \le{}& \int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} \Big(\left|A(s)\right|\left|u(s)\right| + \left|f\left(s,u(s),Ku(s)\right)\right|\Big)\, ds \\
&+ \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)} \Big(\left|A(s)\right|\left|u(s)\right| + \left|f\left(s,u(s),Ku(s)\right)\right| + \left|I_i\left(u\left(t_i^-\right)\right)\right|\Big)\, ds \\
&+ \sum_{i=1}^{k} (t-t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)} \Big(\left|A(s)\right|\left|u(s)\right| + \left|f\left(s,u(s),Ku(s)\right)\right| \\
&+ \left|I_i^*\left(u\left(t_i^-\right)\right)\right|\Big)\, ds \\
&+ \left|\frac{\beta}{\alpha} - t\right| \Bigg[\int_{t_k}^{1} \frac{(1-s)^{q-1}}{\Gamma(q)} \Big(\left|A(s)\right|\left|u(s)\right| + \left|f\left(s,u(s),Ku(s)\right)\right|\Big)\, ds \\
&+ \frac{\beta}{\alpha} \int_{t_k}^{1} \frac{(1-s)^{q-2}}{\Gamma(q-1)} \Big(\left|A(s)\right|\left|u(s)\right| + \left|f\left(s,u(s),Ku(s)\right)\right|\Big)\, ds \\
&+ \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)} \Big(\left|A(s)\right|\left|u(s)\right| + \left|f\left(s,u(s),Ku(s)\right)\right| + \left|I_i\left(u\left(t_i^-\right)\right)\right|\Big)\, ds \\
&+ \sum_{i=1}^{k} \left(\frac{\beta}{\alpha} + 1 - t_k\right) \int_{t_{i-1}}^{t_i} \frac{(t_1-s)^{q-2}}{\Gamma(q-1)} \Big(\left|A(s)\right|\left|u(s)\right| + \left|f\left(s,u(s),Ku(s)\right)\right|
\end{aligned}
$$

$$+ \left|I_i^*\left(u\left(t_i^-\right)\right)\right|\right) ds$$
$$\left. + \frac{1}{\alpha}\left(\left|g_1(u)\right| + \left|g_2(u)\right|\right)\right] + \frac{1}{\alpha}\left|g_1(u)\right|.$$

Thus,

$$\begin{aligned}
\left|(Tu)(t)\right| &\leq (A_1 r + M_1)\int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)}\,ds + (A_1 r + M_1 + M_2)\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}\,ds \\
&\quad + (A_1 r + M_1 + M_3)\sum_{i=1}^{k}|t-t_i|\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}\,ds \\
&\quad + \left(\frac{\beta}{\alpha}+1\right)\left[(A_1 r + M_1)\left(\int_{t_k}^{1}\frac{(1-s)^{q-1}}{\Gamma(q)}\,ds + \frac{\beta}{\alpha}\int_{t_k}^{1}\frac{(1-s)^{q-2}}{\Gamma(q-1)}\,ds\right)\right. \\
&\quad + (A_1 r + M_1 + M_2)\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}\,ds \\
&\quad + (A_1 r + M_1 + M_3)\sum_{i=1}^{k}\left(\frac{\beta}{\alpha}+1\right)\int_{t_{i-1}}^{t_i}\frac{(t_1-s)^{q-2}}{\Gamma(q-1)}\,ds \\
&\quad \left. + \frac{1}{\alpha}(G_1 + G_2)\right] + \frac{1}{\alpha}G_1 \\
&\leq \frac{1}{\Gamma(q+1)}\left(\frac{\beta}{\alpha}+2\right)\left((A_1 r + M_1)(1+p) + pM_2\right) \\
&\quad + \frac{1}{\Gamma(q)}\left[p(A_1 r + M_1 + M_3)\left(1 + \left(\frac{\beta}{\alpha}+1\right)^2\right) + (A_1 r + M_1)\frac{\beta}{\alpha}\left(\frac{\beta}{\alpha}+1\right)\right] \\
&\quad + \frac{1}{\alpha}\left(\frac{\beta}{\alpha}+1\right)(G_1 + G_2) + G_1 = l.
\end{aligned}$$

Then it follows that $\|Tu\| \leq l$.

Step 3: T maps bounded sets into equicontinuous sets.

Let B_r be a bounded set of $PC^1(J)$ as in Step 2, and let $u \in B_r$. Then, letting $\tau_1, \tau_2 \in J_k$ with $\tau_1 < \tau_2$, $0 \leq k \leq p$, we have

$$\left|(Tu)(\tau_2) - (Tu)(\tau_1)\right| \leq \int_{\tau_1}^{\tau_2}\left|(Tu)'(s)\right| ds \leq \widetilde{l}(\tau_2 - \tau_1),$$

where

$$\begin{aligned}
\left|(Tu)'(t)\right| &\leq \int_{t_k}^{t}\frac{(t-s)^{q-2}}{\Gamma(q-1)}\left(\left|A(s)\right|\left|u(s)\right| + \left|f\left(s,u(s),Ku(s)\right)\right|\right) ds \\
&\quad + \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}\left(\left|A(s)\right|\left|u(s)\right| + \left|f\left(s,u(s),Ku(s)\right)\right| + \left|I_i^*\left(u\left(t_i^-\right)\right)\right|\right) ds \\
&\quad + \int_{t_k}^{1}\frac{(1-s)^{q-1}}{\Gamma(q)}\left(\left|A(s)\right|\left|u(s)\right| + \left|f\left(s,u(s),Ku(s)\right)\right|\right) ds
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\beta}{\alpha}\int_{t_k}^{1}\frac{(1-s)^{q-2}}{\Gamma(q-1)}\left(\left|A(s)\right|\left|u(s)\right| + \left|f\left(s,u(s),Ku(s)\right)\right|\right)ds\\
&+ \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}\left(\left|A(s)\right|\left|u(s)\right| + \left|f\left(s,u(s),Ku(s)\right)\right| + \left|I_i\left(u\left(t_i^-\right)\right)\right|\right)ds\\
&+ \sum_{i=1}^{k}\left(\frac{\beta}{\alpha}+1-t_k\right)\int_{t_{i-1}}^{t_i}\frac{(t_1-s)^{q-2}}{\Gamma(q-1)}\left(\left|A(s)\right|\left|u(s)\right| + \left|f\left(s,u(s),Ku(s)\right)\right|\right.\\
&\left.+ \left|I_i^*\left(u\left(t_i^-\right)\right)\right|\right)ds\\
&+ \frac{1}{\alpha}\left(\left|g_1(u)\right| + \left|g_2(u)\right|\right)
\end{aligned}
$$

$$
\begin{aligned}
\left|(Tu)'(t)\right| &\le (A_1r+M_1)\left[\int_{t_k}^{t}\frac{(t-s)^{q-2}}{\Gamma(q-1)}\,ds + \int_{t_k}^{1}\frac{(1-s)^{q-1}}{\Gamma(q)}\,ds + \frac{\beta}{\alpha}\int_{t_k}^{1}\frac{(1-s)^{q-2}}{\Gamma(q-1)}\,ds\right]\\
&\quad+ (A_1r+M_1+M_2)\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}\,ds\\
&\quad+ (A_1r+M_1+M_3)\\
&\quad\times\left[\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}\,ds + \sum_{i=1}^{k}\left(\frac{\beta}{\alpha}+1\right)\int_{t_{i-1}}^{t_i}\frac{(t_1-s)^{q-2}}{\Gamma(q-1)}\,ds\right]\\
&\quad+ \frac{1}{\alpha}(G_1+G_2)\\
&\le \frac{1}{\Gamma(q+1)}\left[2(A_1r+M_1)+M_2\right] + \frac{1}{\alpha}(G_1+G_2)\\
&\quad+ \frac{1}{\Gamma(q)}\left[(A_1r+M_1)\left(\frac{\beta}{\alpha}+1\right) + p(A_1r+M_1+M_3)\left(\frac{\beta}{\alpha}+2\right)\right]\\
&:= \widetilde{l}\quad \text{for any } t\in J_k, 0\le k\le p.
\end{aligned}
$$

Hence, $T(B_r)$ is equicontinuous on all the subintervals J_k, $k = 0,1,2,\dots,p$. Then we can deduce that $T : PC^1(J,X) \to PC^1(J,X)$ is completely continuous as a result of the Arzela-Ascoli theorem together with Steps 1 to 3.

As a consequence of Schauder's fixed point theorem, we conclude that T has a fixed point. That is, BVP (1.1) has at least one solution. The proof is complete. □

Our second result is about the uniqueness of the solution of BVP (1.1). And it depends on Banach's fixed point theorem.

Theorem 2 *Assume that* (A1)-(A8) *hold with*

$$
\begin{aligned}
&\left\{\frac{1}{\Gamma(q+1)}\left(\frac{\beta}{\alpha}+2\right)\left(\left(A_1+L_1(1+L_2)\right)(1+p)+pL_3\right)\right.\\
&+ \frac{1}{\Gamma(q)}\left[p\left(A_1+L_1(1+L_2)+L_4\right)\left(1+\left(\frac{\beta}{\alpha}+1\right)^2\right) + \left(A_1+L_1(1+L_2)\right)\frac{\beta}{\alpha}\left(\frac{\beta}{\alpha}+1\right)\right]\\
&\left.+ \frac{1}{\alpha}\left(\frac{\beta}{\alpha}+1\right)(b_1+b_2)+b_1\right\} := \Omega_{A_1,L_1,L_2,L_3,L_4,b_1,b_2,q,\alpha,\beta} < 1. \qquad (3.2)
\end{aligned}
$$

Proof First, we show that $TB_r \subset B_r$. Indeed, in order to do this, it is adequate to replace l with r in Step 2 in Theorem 1. Thus, T maps B_r into itself. Now, define the mapping $T: C(J, B_r) \to C(J, B_r)$. Then, for each $t \in J$, we have

$$
\begin{aligned}
&\bigl|(Tu)(t) - (Tv)(t)\bigr| \\
&\quad \le \int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} \bigl(\bigl|A(s)\bigr|\bigl|u(s)-v(s)\bigr| \\
&\qquad + \bigl|f\bigl(s,u(s),Ku(s)\bigr) - f\bigl(s,v(s),Kv(s)\bigr)\bigr|\bigr)\,ds + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)} \bigl(\bigl|A(s)\bigr|\bigl|u(s)-v(s)\bigr| \\
&\qquad + \bigl|f\bigl(s,u(s),Ku(s)\bigr) - f\bigl(s,v(s),Kv(s)\bigr)\bigr| + \bigl|I_i\bigl(u(t_i^-)\bigr) - I_i\bigl(v(t_i^-)\bigr)\bigr|\bigr)\,ds \\
&\qquad \times \sum_{i=1}^{k} (t-t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)} \bigl(\bigl|A(s)\bigr|\bigl|u(s)-v(s)\bigr| \\
&\qquad + \bigl|f\bigl(s,u(s),Ku(s)\bigr) - f\bigl(s,v(s),Kv(s)\bigr)\bigr| + \bigl|I_i^*\bigl(u(t_i^-)\bigr) - I_i^*\bigl(v(t_i^-)\bigr)\bigr|\bigr)\,ds \\
&\qquad + \left|\frac{\beta}{\alpha} - t\right| \Biggl[\int_{t_k}^{1} \frac{(1-s)^{q-1}}{\Gamma(q)} \bigl(\bigl|A(s)\bigr|\bigl|u(s)-v(s)\bigr| \\
&\qquad + \bigl|f\bigl(s,u(s),Ku(s)\bigr) - f\bigl(s,v(s),Kv(s)\bigr)\bigr|\bigr)\,ds \\
&\qquad + \frac{\beta}{\alpha} \int_{t_k}^{1} \frac{(1-s)^{q-2}}{\Gamma(q-1)} \bigl(\bigl|A(s)\bigr|\bigl|u(s)-v(s)\bigr| + \bigl|f\bigl(s,u(s),Ku(s)\bigr) - f\bigl(s,v(s),Kv(s)\bigr)\bigr|\bigr)\,ds \\
&\qquad + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)} \bigl(\bigl|A(s)\bigr|\bigl|u(s)-v(s)\bigr| \\
&\qquad + \bigl|f\bigl(s,u(s),Ku(s)\bigr) - f\bigl(s,v(s),Kv(s)\bigr)\bigr| + \bigl|I_i\bigl(u(t_i^-)\bigr) - I_i\bigl(v(t_i^-)\bigr)\bigr|\bigr)\,ds \\
&\qquad + \sum_{i=1}^{k} \left(\frac{\beta}{\alpha} + 1 - t_k\right) \int_{t_{i-1}}^{t_i} \frac{(t_1-s)^{q-2}}{\Gamma(q-1)} \bigl(\bigl|A(s)\bigr|\bigl|u(s)-v(s)\bigr| \\
&\qquad + \bigl|f\bigl(s,u(s),Ku(s)\bigr) - f\bigl(s,v(s),Kv(s)\bigr)\bigr| + \bigl|I_i^*\bigl(u(t_i^-)\bigr) - I_i^*\bigl(v(t_i^-)\bigr)\bigr|\bigr)\,ds \\
&\qquad + \frac{1}{\alpha}\bigl(\bigl|g_1(u) - g_1(v)\bigr| + \bigl|g_2(u) - g_2(v)\bigr|\bigr) \Biggr] + \frac{1}{\alpha}\bigl|g_1(u) - g_1(v)\bigr|.
\end{aligned}
$$

Observing the inequality

$$
\begin{aligned}
\bigl|f\bigl(s,u(s),Ku(s)\bigr) - f\bigl(s,v(s),Kv(s)\bigr)\bigr| &\le L_1\bigl(\bigl|u(s)-v(s)\bigr| + \bigl|Ku(s) - Kv(s)\bigr|\bigr) \\
&\le L_1(1+L_2)\bigl|u(s)-v(s)\bigr|,
\end{aligned}
$$

we have

$$
\begin{aligned}
&\bigl|(Tu)(t) - (Tv)(t)\bigr| \\
&\quad \le \bigl(A_1 + L_1(1+L_2)\bigr) \int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} \bigl|u(s)-v(s)\bigr|\,ds \\
&\qquad + \bigl(A_1 + L_1(1+L_2) + L_3\bigr) \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)} \bigl|u(s)-v(s)\bigr|\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+\bigl(A_1+L_1(1+L_2)+L_4\bigr)\sum_{i=1}^{k}|t-t_i|\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}\bigl|u(s)-v(s)\bigr|\,ds\\
&+\left(\frac{\beta}{\alpha}+1\right)\left[\bigl(A_1+L_1(1+L_2)\bigr)\left(\int_{t_k}^{1}\frac{(1-s)^{q-1}}{\Gamma(q)}\bigl|u(s)-v(s)\bigr|\,ds\right.\right.\\
&+\left.\frac{\beta}{\alpha}\int_{t_k}^{1}\frac{(1-s)^{q-2}}{\Gamma(q-1)}\bigl|u(s)-v(s)\bigr|\,ds\right)\\
&+\bigl(A_1+L_1(1+L_2)+L_3\bigr)\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}\bigl|u(s)-v(s)\bigr|\,ds\\
&+\bigl(A_1+L_1(1+L_2)+L_4\bigr)\sum_{i=1}^{k}\left(\frac{\beta}{\alpha}+1\right)\int_{t_{i-1}}^{t_i}\frac{(t_1-s)^{q-2}}{\Gamma(q-1)}\bigl|u(s)-v(s)\bigr|\,ds\\
&+\left.\frac{1}{\alpha}(b_1+b_2)\bigl|u(t)-v(t)\bigr|\right]+\frac{1}{\alpha}b_1\bigl|u(t)-v(t)\bigr|.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\bigl\|(Tu)(t)-(Tv)(t)\bigr\|\\
&\quad\le\left\{\frac{1}{\Gamma(q+1)}\left(\frac{\beta}{\alpha}+2\right)\bigl(\bigl(A_1+L_1(1+L_2)\bigr)(1+p)+pL_3\bigr)\right.\\
&\qquad+\frac{1}{\Gamma(q)}\left[p\bigl(A_1+L_1(1+L_2)+L_4\bigr)\left(1+\left(\frac{\beta}{\alpha}+1\right)^2\right)+\bigl(A_1+L_1(1+L_2)\bigr)\frac{\beta}{\alpha}\left(\frac{\beta}{\alpha}+1\right)\right]\\
&\qquad+\left.\frac{1}{\alpha}\left(\frac{\beta}{\alpha}+1\right)(b_1+b_2)+b_1\right\}\|u-v\|,
\end{aligned}
$$

which implies that

$$
\bigl\|(Tu)(t)-(Tv)(t)\bigr\|\le\Omega_{A_1,L_1,L_2,L_3,L_4,b_1,b_2,q,\alpha,\beta}\|u-v\|.
$$

Therefore, by (3.2), the operator T is a contraction. As a consequence of Banach's fixed point theorem, we deduce that T has a fixed point which is a unique solution of BVP (1.1). □

Example 1 Consider the following boundary value problem for impulsive integrodifferential evolution equation of fractional order:

$$
\begin{aligned}
&{}^{C}D^{\frac{3}{2}}u(t)=\frac{1}{20}\bigl(\cos^2 t\bigr)u(t)+\frac{(\sin 7t)|u(t)|}{(t+5)^4(1+|u(t)|)}\\
&\qquad\qquad+\int_0^t e^{-\frac{1}{25}u(s)}\,ds,\quad t\in[0,1], t\neq\frac{1}{2},\\
&\Delta u\left(\frac{1}{2}\right)=\frac{|u(\frac{1}{2}^-)|}{15+|u(\frac{1}{2}^-)|},\qquad \Delta u'\left(\frac{1}{2}\right)=\frac{|u'(\frac{1}{2}^-)|}{10+|u'(\frac{1}{2}^-)|}\\
&3u(0)+u'(0)=\sum_{i=1}^{m}\eta_i u(\xi_i),\qquad 3u(1)+u'(1)=\sum_{j=1}^{m}\widetilde{\eta}_j\widetilde{u}(\xi_i),
\end{aligned}
\tag{3.3}
$$

where $0 < \eta_1 < \eta_2 < \cdots < 1$, $0 < \widetilde{\eta}_1 < \widetilde{\eta}_2 < \cdots < 1$, and $\eta_i, \widetilde{\eta}_j$ are given positive constants with $\sum_{i=1}^{m} \eta_i < \frac{2}{15}$ and $\sum_{j=1}^{m} \widetilde{\eta}_j < \frac{3}{15}$.

Here, $\alpha = 3$, $\beta = 1$, $q = \frac{3}{2}$, $p = 1$. Obviously, $A_1 = \frac{1}{10}$, $L_1 = \frac{1}{125}$, $L_2 = \frac{1}{25}$, $L_3 = \frac{1}{15}$, $L_4 = \frac{1}{10}$, $b_1 = \frac{2}{15}$, $b_2 = \frac{3}{15}$ and by (2.5), it can be found that

$$\Omega_{A_1,L_1,L_2,L_3,L_4,b_1,b_2,q,\alpha,\beta} = \frac{1,203,709}{843,750\sqrt{\pi}} + \frac{38}{135} = 0.63361 < 1.$$

Therefore, due to the fact that all the assumptions of Theorem 2 hold, BVP (3.3) has a unique solution. Besides, one can easily check the result of Theorem (1) for BVP (3.3).

Conclusion

In the literature, the authors consider impulsive fractional semilinear evolution integro-differential equations of order $0 < q < 1$ in different aspects as mentioned above. Besides, either impulsive fractional semilinear equations of order $1 < q < 2$ or impulsive fractional integro-differential equations of order $1 < q < 2$ are studied by different authors (see, for instance, [44, 45]). But, to the best of our knowledge, no study considering both cases has been carried out. Thus, in this article, we consider a general boundary value problem for impulsive fractional semilinear evolution integro-differential equations of order $1 < q < 2$ with nonlocal conditions. Therefore, the present results are new and complementary to previously known literature.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final draft.

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